Resonances and entrainment breakup in Kuramoto models with multimodal frequency densities

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We characterize some intriguing aspects of the entrainment behavior of coupled oscillators by means of a perturbation analysis of the partially synchronized solution of the classical Kuramoto-Sakaguchi model. The analysis reveals that partial entrainment may disappear with increasing coupling strength. It also predicts the occurrence of resonances: partial entrainment is induced in oscillators with natural frequencies in specific intervals not corresponding to high oscillator densities. The results are illustrated by simulations.

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I. INTRODUCTION

A popular model for the study of synchronization of systems of coupled oscillators is the Kuramoto model [1]. Several examples, including flashing fireflies, coupled laser arrays, and pacemaker cells in the heart, are described in [2]. In [3] the model was extended to the system described by

$$\dot{\theta}_i(t) = \omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin[\theta_j(t) - \theta_i(t) - \alpha], \quad \forall\, t \in \mathbb{R}, \quad (1)$$

for all i in $\{1,\ldots,N\}$, where N is the number of oscillators in the system, $K \geq 0$ is the coupling strength, $|\alpha| \leq \frac{\pi}{2}$, and the ω_i values are real numbers drawn from a given distribution. The parameters ω_i represent the natural frequencies of the oscillators and determine the behavior of the system for $K=0$. Kuramoto and Sakaguchi [3] considered the limit $N \to \infty$ and showed that for K larger than a critical value K_c there is a solution that exhibits partial synchronization. This solution is characterized by two different groups of oscillators; those in the first group are locked at some frequency Ω_0 while the remaining oscillators are moving with long-term average frequencies different from Ω_0. The stability properties of this solution are not yet fully understood.

Partial synchronization and partial entrainment (which we define later on) are specific types of clustering that can be observed in systems of coupled oscillators. In general, in systems of coupled oscillators, one distinguishes between *phase* clustering and *frequency* clustering. The first form (see, e.g., [4–6]) is observed in a network of oscillators with identical natural frequencies. The clusters consist of oscillators with equal phases, while oscillators from different clusters have different phases. If the natural frequencies are not identical but their differences are sufficiently small, then it may still be possible to distinguish different phase clusters [7]; this phenomenon is also related to *multibranch entrainment* [8]. Larger differences between the natural frequencies may induce oscillators having different long-term average frequencies, resulting in frequency clustering [9]: each cluster is characterized by the long-term average frequency of its members. This type of clustering is characteristic for the Kuramoto model.

Although research on the Kuramoto model seems to focus on the case of a finite number of oscillators [10–17], the model with an infinite number of oscillators continues to enjoy attention, and analytical results have been obtained for, e.g., the noisy model [18,19], the periodically driven model [20,21], and the model with general periodic interaction functions [22]. Multimodal frequency distributions have already been considered in [19,23] (with the distribution limited to be discrete—i.e., to be a sum of Dirac impulses) and [24] [where two interacting populations are considered, and for equal intra- and interpopulation coupling strength the model (1) with a bimodal distribution is retrieved], with the focus on stability regions for the different types of solutions. For overviews on the Kuramoto model we refer to [25–27].

In this paper we consider a perturbation of the solution of the classical model discussed in [3] (i.e., with an infinite number of oscillators), arising from a change in the unimodal density function. The change consists of adding a small fraction of oscillators with natural frequencies in small intervals, corresponding to some narrow extra peaks in the density function, resulting in a multimodal distribution of the natural frequencies. For appropriate values of the coupling strength K additional entrained subsets of oscillators will appear. A first-order analysis with respect to their size shows that these sets can be characterized analytically.

In our main result we provide analytical evidence that the multimodal distribution allows the occurrence of two phenomena which cannot be observed in the classical case [3]. For a particular choice of the frequency distribution we show analytically that the extra entrained subsets may disappear when the coupling strength is increased, a phenomenon that may also be observed in systems with a finite number of oscillators [28]. The analysis also predicts the occurrence of resonances: the extra entrained subsets may induce entrainment in several other subsets of oscillators, which are not necessarily associated with high frequency densities. The second phenomenon is akin to the occurrence of Shapiro steps in the periodically driven Kuramoto model [20,21,29]. Up to now these phenomena have been observed in simulations; see, e.g., [27], pp. 46–47, for simulations regarding both phenomena for the case $\alpha=0$, and [24] for simulations regarding the occurrence of resonances. In the present paper we provide analytical results, which fully characterize the

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entrained subsets and explain these phenomena.

In the next section we introduce the order parameter r, written in terms of the oscillator density ρ. In the auxiliary material [30] a particular solution for the density ρ is expressed in terms of r; this leads to a self-consistency equation for r, for which the results are discussed in Sec. III. We conclude with two numerical examples, illustrating the aforementioned phenomena.

II. PRELIMINARIES

The distribution of the natural frequencies is characterized by a continuous normalized density function g. When discussing the model with an infinite number of oscillators, we will consider sets S of oscillators which can be associated with a countable union of disjunct intervals $L_k \subset \mathbb{R}$ ($k \in \{1,\ldots,\mathcal{N}\}$, with $\mathcal{N} \in \mathbb{N} \cup \{+\infty\}$), such that S contains all oscillators with a natural frequency in one of the intervals L_k. The sum $\sum_{k\in I}\int_{L_k} g(\omega)d\omega$ will be called the *weight* of S, and represents the fraction of the population contained in S.

A subset with nonzero weight also has a nonzero contribution in the interaction with other oscillators, and in the following definition of partial entrainment we demand that entrained subsets have nonzero weight to ensure that the entrainment can be (at least partially) attributed to the interaction between the oscillators in the entrained subset.

A solution is said to exhibit *partial entrainment* with respect to a subset S_e of oscillators if S_e has nonzero weight and if the phase difference between any pair of oscillators in S_e is *bounded*. The set S_e will be called a(n) (partially) *entrained subset*, and its members are called *entrained oscillators*.

Because of the continuity of g and the definition of partial entrainment, an entrained subset contains at least two oscillators with different natural frequencies. From the system equations one can easily derive the result that, if the phase difference of two oscillators is bounded, then the phase difference of one of these oscillators with any other oscillator with a natural frequency in between their natural frequencies is also bounded. With a maximal entrained subset (i.e., an entrained subset that is not included in a larger entrained subset) we can therefore associate an interval (with nonzero length) such that the entrained subset contains all oscillators with natural frequencies in this interval. The set containing the frequencies of all entrained oscillators will be denoted by $\mathcal{E}$. It is a countable union of intervals.

The nonentrained oscillators will be referred to as *drifting oscillators*, in accordance with the use of the term in [25]. If the natural frequency ω of a drifting oscillator satisfies $g(\omega) > 0$, then the oscillator has unbounded phase differences with any other oscillator with a different natural frequency.

In systems with a finite number of oscillators the weight of an arbitrary set of oscillators is naturally defined as the ratio of the number of oscillators it contains and the number of oscillators in the system. The definition of partial entrainment remains applicable to systems with a finite number of oscillators, and is in accordance with the definition in [28]. The definition still leads to different interpretations depending on whether the number of oscillators is finite or infinite when considering two oscillators with equal natural frequencies. Since two oscillators in a finite population always constitute a nonzero fraction of the population, two oscillators with equal natural frequencies (which consequently have bounded phase differences) are considered to be entrained. As they constitute a nonzero fraction of the population, the interaction between the two oscillators cannot be neglected, and a small frequency shift, resulting in different natural frequencies, will not break up the entrainment. In an infinite population, however, any difference in natural frequencies (no matter how small) of two drifting oscillators in general results in unbounded phase differences. A group of oscillators with a common natural frequency is insufficiently large (as the group constitutes a zero fraction of the population) to be able to influence and entrain oscillators with a different natural frequency.

We consider a perturbation of the solution investigated by Kuramoto and Sakaguchi. The unperturbed solution involves one partially synchronized group of oscillators (i.e., partially entrained but with constant mutual phase differences) moving at a constant frequency Ω_0, while the remaining oscillators are moving with average frequencies different from Ω_0. The perturbation causes a (small) fraction of the drifting oscillators to become entrained.

For the perturbed solution, define Ω_0 as the long-term average frequency of the largest entrained subset (the one that corresponds to the synchronized subset for the unperturbed solution). We switch to a rotating frame (with constant frequency Ω_0) by replacing θ_i by $\theta_i + \Omega_0 t$. At the same time we replace ω_i by $\omega_i + \Omega_0$, leaving the system equations unchanged. The new ω_i values can be considered as drawn from the density function $\tilde{g}$, with $\tilde{g}(\omega) = g(\Omega_0 + \omega)$, where g is the original density function. In the rotating frame, the largest entrained subset is moving on average at zero frequency, making it possible to look for an almost stationary solution. For the remainder of this paper, θ_i values are to be considered with respect to this rotating frame, and ω_i values are considered to be drawn from the density function $\tilde{g}$.

Defining the real-valued functions r and ψ by

$$r(t)e^{i\psi(t)} = \frac{1}{N}\sum_{j=1}^{N} e^{i[\theta_j(t) - C_0]}, \quad \forall\, t \in \mathbb{R} \tag{2}$$

$[r(t) \geq 0]$, where $C_0 \in \mathbb{R}$ will be determined later, the system equations can be rewritten as

$$\dot{\theta}_i(t) = \omega_i + Kr(t)\sin[\psi(t) + C_0 - \theta_i(t) - \alpha], \tag{3}$$

for all t in $\mathbb{R}$, for all i in $\{1,\ldots,N\}$.

The parameter $r(t)$ can be seen as an order parameter, since when all oscillators are close together $r(t)$ will be close to 1, and when they are spread uniformly over the interval $[-\pi,\pi]$, $r(t)$ will be zero. Since we consider a perturbation of a stationary solution we set $r(t)e^{i\psi(t)} = [r_0 + r'(t)]e^{i\psi_0}$, where $r_0 e^{i\psi_0}$ (with $r_0 \in \mathbb{R}_0^+$) equals the long-term average value of $r(t)e^{i\psi(t)}$ (which we assume to exist), and the complex-valued perturbation r' has real part r'_R and imaginary part r'_I. Setting $C_0 \triangleq \alpha - \psi_0$, we rewrite (3) as

$$\dot{\theta}_i(t) = \omega_i - K[r_0 + r'_R(t)]\sin\,\theta_i(t) + Kr'_I(t)\cos\,\theta_i(t), \quad (4)$$

for all t in $\mathbb{R}$, and (2) as

$$r_0 + r'_R(t) + ir'_I(t) = \frac{1}{N}\sum_{j=1}^{N} e^{i(\theta_j(t)-\alpha)}, \quad \forall\, t \in \mathbb{R}. \quad (5)$$

The model with infinite N can be described in terms of a population density $\rho(\theta,\omega,t)$, which we consider to be periodic in θ with period 2π; the fraction of the population of oscillators with a natural frequency equal to ω and a phase value in $\cup_{m\in\mathbb{Z}}[\theta+2\pi m, \theta+d\theta+2\pi m]$ at time t equals $\rho(\theta,\omega,t)d\theta$ for infinitesimally small values of $d\theta$. This implies that $\int_{-\pi}^{\pi}\rho(\theta,\omega,t)d\theta=1$, $\forall \omega, t\in\mathbb{R}$. Equation (5) then becomes

$$r_0 + r'_R(t) + ir'_I(t) = e^{-i\alpha}\int_{-\infty}^{\infty} d\omega\; \tilde{g}(\omega)\int_{-\pi}^{\pi} d\theta\; \rho(\theta,\omega,t)e^{i\theta}, \quad (6)$$

for all t in $\mathbb{R}$.

Letting $v(\theta,\omega,t)$ represent the frequency of an oscillator with natural frequency ω and phase θ at time t, the evolution of ρ is determined by the continuity equation

$$\frac{\partial\rho}{\partial t} = -\frac{\partial(\rho v)}{\partial\theta}, \quad (7)$$

where $v(\theta,\omega,t)=\omega-K[r_0+r'_R(t)]\sin\,\theta+Kr'_I(t)\cos\,\theta$, for all θ, ω, t in $\mathbb{R}$. We assume that r'_R, r'_I, and r' can be represented by Fourier sums as follows:

$$r'_R(t) = \sum_{\gamma\in\Gamma} R'_R(\gamma)e^{i\gamma t}, \quad r'_I(t) = \sum_{\gamma\in\Gamma} R'_I(\gamma)e^{i\gamma t},$$

$$r'(t) = \sum_{\gamma\in\Gamma} R'(\gamma)e^{i\gamma t} = \sum_{\gamma\in\Gamma}[R'_R(\gamma) + iR'_I(\gamma)]e^{i\gamma t},$$

for all t in $\mathbb{R}$, where $|R'_R|^2+|R'_I|^2:\mathbb{R}\to\mathbb{R}$ is zero everywhere, except on the set $\Gamma\in\mathbb{R}$, for which we assume that $\Gamma\cap[a,b]$ is a finite set, for any $a,b\in\mathbb{R}$ with $a<b$. Furthermore, we can assume that $0\notin\Gamma$ because of the definition of r', and since r'_R and r'_I are real-valued functions, $\gamma\in\Gamma$ implies $-\gamma\in\Gamma$. We will find a self-consistency equation for $r'(t)$ by expressing the right-hand side of (6) up to first order in $r'_{\text{rms}}\in\mathbb{R}^+$, with $r'^2_{\text{rms}}\triangleq\lim_{t\to\infty}\frac{1}{t}\int_0^t|r'(t')|^2dt'=\Sigma_{\gamma\in\Gamma}|R'(\gamma)|^2$; r'_{rms} is assumed to be bounded and small with respect to r_0 and $\min_{\gamma\in\Gamma}\frac{|\gamma|}{K}$. As will be shown in the analysis, each exponential component of r' with frequency γ will result in an entrained subset of oscillators moving at average frequency γ.

III. RESULTS

Based on Eqs. (6) and (7), we obtain a self-consistency equation, determining Ω_0, r_0, the set Γ, and $R'(\gamma)$ for any $\gamma\in\Gamma$. For the details we refer to the auxiliary material [30]; in this section we will discuss the results of these calculations.

While the oscillators in the largest entrained subset have natural frequencies in the interval $[-Kr_0, Kr_0]$, any other entrained subset corresponds to an interval

$$[\omega - K|R'_C(\gamma_\omega)|, \omega + K|R'_C(\gamma_\omega)|],$$

with $R'_C(\gamma_\omega)\triangleq R'_R(\gamma_\omega)+i\frac{\gamma_\omega}{\omega}R'_I(\gamma_\omega)$, and where $\omega\in\mathbb{R}\setminus[-Kr_0,Kr_0]$ is such that $\gamma_\omega\triangleq\omega\sqrt{1-(\frac{Kr_0}{\omega})^2}\in\Gamma$. The oscillators in this subset are moving with a time-averaged frequency of γ_ω.

The calculations in [30] eventually lead to the following conditions:

$$r_0e^{i\alpha} = Kr_0\int_{-\pi/2}^{\pi/2} d\theta_0\,\cos\,\theta_0\tilde{g}(Kr_0\,\sin\,\theta_0)e^{i\theta_0} + \sum_{\gamma_{\omega'}\in\Gamma}\int_{-\pi/2}^{\pi/2} d\theta_0 i(\omega'-\gamma_{\omega'})\frac{|R'_C(\gamma_{\omega'})|}{r_0}\cos\,\theta_0\tilde{g}[\omega'+K|R'_C(\gamma_{\omega'})|\sin\,\theta_0]$$
$$+ i\int_{\mathbb{R}\setminus\mathcal{E}} d\omega\;\tilde{g}(\omega)\frac{\omega-\gamma_\omega}{Kr_0},$$

$$R'(\gamma_{\omega'})e^{i\alpha} = 2iK^2r_0\int_{-\pi/2}^{\pi/2} d\theta_0\,\cos\,\theta_0\tilde{g}(Kr_0\,\sin\,\theta_0)e^{i\theta_0}\frac{Kr_0\cos\,\theta_0 - i\gamma_{\omega'}}{2(K^2r_0^2\cos^2\,\theta_0+\gamma^2_{\omega'})}[\cos\,\theta_0R'_I(\gamma_{\omega'}) - \sin\,\theta_0R'_R(\gamma_{\omega'})]$$
$$+ \sum_{\gamma_{\omega''}\in\Gamma\cap\left\{\frac{\gamma_{\omega'}}{k}:k\in N_0\right\}}\int_{-\pi/2}^{\pi/2} d\theta_0\frac{|R'_C(\gamma_{\omega''})|}{r_0}\cos\,\theta_0\tilde{g}[\omega''+K|R'_C(\gamma_{\omega''})|\sin\,\theta_0](-2i)\gamma_{\omega''}\left(ie^{i\{\theta_0+\arg[R'_C(\gamma_{\omega''})]\}}\frac{\omega''-\gamma_{\omega''}}{Kr_0}\right)^{\gamma_{\omega'}/\gamma_{\omega''}}$$
$$+ i\int_{\mathbb{R}\setminus\mathcal{E}} d\omega\;\tilde{g}(\omega)\frac{\omega-\gamma_{\omega''}}{Kr_0^2(\gamma_-\gamma_{\omega'})}[\omega R'_R(\gamma_{\omega'}) + i\gamma_{R'_I}(\gamma_{\omega'})],$$

for all $\gamma_{\omega'}$ in Γ. In both equations the right-hand side consists of three contributions, corresponding to the entrained oscillators in the largest entrained subset with average frequency Ω_0, the entrained oscillators in the smaller entrained subsets with average frequencies $\gamma\in\Gamma$, and the drifting oscillators. There is always the solution $r_0=0$ and $R'(\gamma)=0$, $\forall\ \gamma\in\mathbb{R}$, which corresponds to total incoherence with every oscillator moving at its natural frequency ω. For sufficiently large values of the coupling strength, other solutions may arise, as we describe in Secs. III A and III B.

Remark 1. For $\alpha=0$ and g even, performing the substitutions $\omega\leftrightarrow-\omega$, $\theta\leftrightarrow-\theta,\ldots$, and $R_I'\leftrightarrow-R_I'$ in the summations and integrals [in the equations for $R'(\gamma_{\omega'})$], followed by complex conjugation, corresponds to interchanging the equations for $R'(\gamma_{\omega'})$ and $R'(-\gamma_{\omega'})$. It then follows that R_I' (and thus also r_I') can be put equal to zero everywhere, while for each positive $\gamma_{\omega'}$ only one (complex) equation remains. This does not simplify the corresponding expressions, but the number of unknown variables is reduced.

A. The solution by Kuramoto and Sakaguchi

If K is larger than some critical value K_c there is also a nonzero solution for r_0, corresponding to an entrained subset of oscillators with ω values in $[-Kr_0,Kr_0]$. This solution was studied in [3]. The zeroth-order values for r_0 and Ω_0 can be deduced from the equations

$$K\int_{-\pi/2}^{\pi/2}\cos^2\theta_0 d\theta_0\tilde{g}(Kr_0\sin\theta_0)=\cos\alpha, \tag{8a}$$

$$Kr_0\int_{-\pi/2}^{\pi/2}\sin\theta_0\cos\theta_0 d\theta_0\tilde{g}(Kr_0\sin\theta_0) + \int_{\mathbb{R}\setminus[-Kr_0,Kr_0]} d\omega\,\tilde{g}(\omega)\frac{\omega-\gamma_\omega}{Kr_0}=r_0\sin\alpha, \tag{8b}$$

and we retrieve the results from [3], where a single entrained subset was considered. Notice that the values of r_0 and Ω_0 derived from the above equations are exact only if no other entrained subsets are present (and consequently $\Gamma=\emptyset$).

For K equal to K_c, at the onset of the entrainment, r_0 equals 0 and it can be derived that (in zeroth order for r_0 and Ω_0)

$$\frac{\pi}{2}K_c g(\Omega_0)=\cos\alpha,$$

$$K_c\int_0^\infty\frac{g(\Omega_0+\omega)-g(\Omega_0-\omega)}{2\omega}d\omega=\sin\alpha.$$

For $\alpha=0$ the second equation determines Ω_0, and then K_c follows immediately from the first equation. If, furthermore, g is even and unimodal, it easily follows that $\Omega_0=0$, and $K_c=\frac{2\cos\alpha}{\pi g(0)}=\frac{2\cos\alpha}{\pi\max(g)}$. For general α and g (but g having a finite number of maxima) it can be easily seen that for small Ω_0 the left-hand side of the second equation will be positive, while for large Ω_0 it will be negative. It follows that for $|\alpha|$ small enough there always exists a solution Ω_0 for this equation. The coupling strength K needs to be at least $\frac{2\cos\alpha}{\pi\max(g)}$ for the existence of an entrained subset, and consequently this value constitutes a lower bound for K_c.

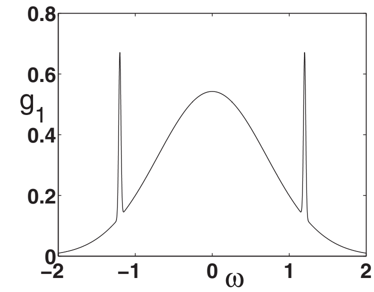

FIG. 1. Density function g_1.

B. Resonances

A nonzero solution for $R'(\gamma_{\omega'})$, for some $\gamma_{\omega'}\in\mathbb{R}$, corresponds to another entrained subset of oscillators with natural frequencies in

$$[\omega'-K|R_C'(\gamma_{\omega'})|,\omega'+K|R_C'(\gamma_{\omega'})|]$$

and moving at an average frequency equal to $\gamma_{\omega'}$. The entrainment of this subset again requires K to be sufficiently large, but this may be easier to see from the formulation in Sec. III D, where a specific class of density functions g is considered and α is set equal to zero. However, the dependence on r_0 cannot be neglected, resulting in more complicated equations than (8a) and (8b).

In general, the entrained subset corresponding to $R'(\gamma_{\omega'})$ will induce other entrained subsets at frequencies $-\gamma_{\omega'}$ and $k\gamma_{\omega'}$, with $k\in\mathbb{N}_0$, as follows from the corresponding equation. This will lead to entrainment at all frequencies $k\gamma_{\omega'}$, with $k\in\mathbb{Z}_0$, in a similar way as for the periodically driven Kuramoto model [20,21].

The basic underlying principle can be summarized as follows. Due to the presence of several entrained subsets induced by high frequency densities, these entrained subsets do not move uniformly, and consequently their contribution to the right-hand side of (6) does not solely consist of the frequency component corresponding to their average frequency, but also contains higher harmonics, which in turn result in entrainment at the corresponding frequencies through Eq. (4).

C. Recursive solution

From now on we assume that g is the sum of a smooth unimodal density function, which does not change abruptly in intervals with a size of the order $r'_{\rm rms}$, and a distribution with one (or two if g is even) narrow peak(s) (with width of the order of $r'_{\rm rms}$), with (a) maximum value(s) of the same order of magnitude as for the first (smooth) contribution. (An example is presented in Fig. 1.) Then we can set $\Gamma=\{k\gamma_{\tilde{\omega}}:k\in\mathbb{Z}_0\}$, for some $\gamma_{\tilde{\omega}}\in\mathbb{R}_0^+$, where $\tilde{\omega}$ or $-\tilde{\omega}$ correspond(s) to the peak(s) in $\tilde{g}$.

The values of r_0 and Ω_0 can be calculated in zeroth order from (8a) and (8b), and using this value for r_0 the equations for $R'(\pm\gamma_{\tilde{\omega}})$ are still correct in lowest (i.e., first) order for

$R'(\pm\gamma_{\tilde{\omega}})$. To solve the equations for $R'(\pm\gamma_{\tilde{\omega}})$ in first order in r'_{rms}, the values of $R'(k'\gamma_{\tilde{\omega}})$ with $|k'|>1$ are not required. Notice that they are needed to characterize the set $\mathcal{E}$ and therefore also $\mathbb{R}\backslash\mathcal{E}$, which appears in one of the integrals, but the integrals over intervals corresponding to $|k'|>1$ will contribute only in second order in r'_{rms}. The contribution for the intervals with $k'=\pm 1$ cannot be neglected, as the denominator in the integrand may become zero, and for the first-order approximation of the entire integral, we can integrate over $\mathbb{R}\backslash\mathcal{E}_1$, where

$$\mathcal{E}_1 \triangleq [-|\tilde{\omega}| - K|R'_C(\gamma_{\tilde{\omega}})|, -|\tilde{\omega}| + K|R'_C(\gamma_{\tilde{\omega}})|] \cup [-Kr_0, Kr_0] \cup [|\tilde{\omega}| - K|R'_C(\gamma_{\tilde{\omega}})|, |\tilde{\omega}| + K|R'_C(\gamma_{\tilde{\omega}})|].$$

The equations for $R'(\pm\gamma_{\tilde{\omega}})$ can then be solved to obtain the value(s) of $\Omega_0 \pm \gamma_{\tilde{\omega}}$ in first order in r'_{rms} (both values if g is even, one of the two if g is not even), together with the moduli of $R'_R(\pm\gamma_{\tilde{\omega}})$ and $R'_I(\pm\gamma_{\tilde{\omega}})$ and their phase difference [if g is not even; if g is even then $R'_{\text{I}}(\pm\gamma_{\tilde{\omega}})=0$]. One degree of freedom remains, as the equations are invariant under a common phase shift for $R'_{\text{R}}(\gamma_{\tilde{\omega}})$ and $R'_{\text{I}}(\gamma_{\tilde{\omega}})$.

The value of $\gamma_{\tilde{\omega}}$ is then known in lowest (i.e., zeroth) order in r'_{rms}. Because of the assumption on g, the values calculated for $\tilde{g}(\omega')$, with $\gamma_{\omega'}=k\gamma_{\tilde{\omega}}$ for $k\in\mathbb{Z}\backslash\{-1,0,1\}$, will also be correct in zeroth order, guaranteeing the correctness of the remaining equations in first order in r'_{rms} for this value of $\gamma_{\tilde{\omega}}$. Since again $R'(k'\gamma_{\tilde{\omega}})$ does not appear in the equation for $R'(k\gamma_{\tilde{\omega}})$ if $|k'|>|k|$, we can progress recursively, calculating the values of $R'_R(k\gamma_{\tilde{\omega}})$ and $R'_I(k\gamma_{\tilde{\omega}})$ in increasing order for $|k|$. At step k the equations for $R'(\pm k\gamma_{\tilde{\omega}})$ are considered, which are two complex equations with two complex unknowns $R'_R(k\gamma_{\tilde{\omega}})$ and $R'_I(k\gamma_{\tilde{\omega}})$. [Recall that $R'_R(-k\gamma_{\tilde{\omega}})=R'_R(k\gamma_{\tilde{\omega}})$ and $R'_I(-k\gamma_{\tilde{\omega}})=R'_I(k\gamma_{\tilde{\omega}})$.] The set $\mathbb{R}\backslash\mathcal{E}$ appearing in one of the integrals can again be replaced by a set that does not depend on $R'_C(k'\gamma_{\tilde{\omega}})$ for $|k'|>|k|$. (More than one choice is possible here.)

D. The symmetric case

For the numerical examples later on we take $\alpha=0$ and g even, reducing the number of unknown variables (see Remark 1):

$$1 = K\int_{-\pi/2}^{\pi/2} g(Kr_0 \sin\theta_0)\cos^2\theta_0 d\theta_0, \tag{9a}$$

$$\Omega_0 = 0, \tag{9b}$$

$$\begin{aligned} 1 = Kr_0 &\int_{-\pi/2}^{\pi/2} d\theta_0 \cos\theta_0 g(Kr_0\sin\theta_0)\frac{K\sin^2\theta_0}{K^2r_0^2\cos^2\theta_0 + k^2\gamma_{\tilde{\omega}}^2}(Kr_0\cos\theta_0 - ik\gamma_{\tilde{\omega}}) \\ &+ \sum_{\substack{k'\in\mathbb{Z}_0:k/k'\in\mathbb{N}_0 \\ \gamma_{\omega'}\triangleq k'\gamma_{\tilde{\omega}}}} \int_{-\pi/2}^{\pi/2} d\theta_0 \frac{|R'_R(k'\gamma_{\tilde{\omega}})|}{r_0R'_R(k\gamma_{\tilde{\omega}})}\cos\theta_0 g[\omega' + K|R'_R(k'\gamma_{\tilde{\omega}})|\sin\theta_0](-2i)\gamma_{\omega'}\left(ie^{i\{\theta_0+\arg[R'_R(k'\gamma_{\tilde{\omega}})]\}}\frac{\omega'-\gamma_{\omega'}}{Kr_0}\right)^{k/k'} \\ &+ i\int_{\mathbb{R}\backslash\mathcal{E}_1} d\omega\, g(\omega)\frac{\omega(\omega-\gamma_\omega)}{Kr_0^2(\gamma_\omega - k\gamma_{\tilde{\omega}})}, \end{aligned} \tag{9c}$$

$$R'_{\text{I}}(k\gamma_{\tilde{\omega}}) = 0, \tag{9d}$$

$\forall\ k\in\mathbb{Z}_0$. From (9a) r_0 can be calculated. In (9c), for $k=1$, the summation reduces to the term corresponding to $k'=1$ (and thus $\omega'=\tilde{\omega}$), and the real and imaginary parts of the corresponding equation can then be rewritten as follows:

$$\begin{aligned} 1 = &\int_{-\pi/2}^{\pi/2} g(Kr_0\sin\theta_0)\frac{K^3r_0^2\sin^2\theta_0\cos^2\theta_0}{K^2r_0^2\cos^2\theta_0 + \gamma_{\tilde{\omega}}^2}d\theta_0 \\ &+ \int_{-\pi/2}^{\pi/2} g[\tilde{\omega} + K|R'_R(\gamma_{\tilde{\omega}})|\sin\theta_0]\frac{2K\gamma_{\tilde{\omega}}\cos^2\theta_0}{\tilde{\omega}+\gamma_{\tilde{\omega}}}d\theta_0, \end{aligned} \tag{10a}$$

$$\begin{aligned} 0 = &-\int_{-\pi/2}^{\pi/2} g(Kr_0\sin\theta_0)\frac{K^2r_0\gamma_{\tilde{\omega}}\cos\theta_0\sin^2\theta_0}{K^2r_0^2\cos^2\theta_0 + \gamma_{\tilde{\omega}}^2}d\theta_0 \\ &+ \int_{-\pi/2}^{\pi/2} g[\tilde{\omega} + K|R'_R(\gamma_{\tilde{\omega}})|\sin\theta_0]\frac{2K\gamma_{\tilde{\omega}}\sin\theta_0\cos\theta_0}{\tilde{\omega}+\gamma_{\tilde{\omega}}}d\theta_0 \\ &+ \int_{\mathbb{R}\backslash\mathcal{E}_1} d\omega\, g(\omega)\frac{\omega(\omega-\gamma_\omega)}{Kr_0^2(\gamma_\omega - \gamma_{\tilde{\omega}})}. \end{aligned} \tag{10b}$$

These equations can be solved numerically to find the values of $\gamma_{\tilde{\omega}}$ and $|R'_R(-\gamma_{\tilde{\omega}})|=|R'_R(\gamma_{\tilde{\omega}})|$.

The value of $R_R'(k\gamma_{\tilde{\omega}})$ with $|k|>1$ is characterized by similar equations, but they become more complicated as k increases, since each equation will include contributions from entrained subsets corresponding to smaller values of k (in absolute value). In general the above equations will have to be solved numerically, as we have done in the examples in Sec. IV. However, we can make the following qualitative observation indicating that entrainment may disappear with increasing coupling strength.

E. Entrainment breakup with increasing K

If we assume that $\tilde{\omega}$ cannot deviate much from a value where a peak of g is attained, then it follows that with increasing K the value of Kr_0 may approach $\tilde{\omega}$ and thus $\gamma_{\tilde{\omega}}$ may approach zero. However, for $\gamma_{\tilde{\omega}}=0$, Eq. (10a) reduces to

$$1 = K\int_{-\pi/2}^{\pi/2} g(Kr_0 \sin \theta_0)\sin^2 \theta_0 d\theta_0.$$

If g is decreasing in $(0,Kr_0)$, then the right-hand side is smaller than the right-hand side of (9a) (equaling 1), and therefore this equation cannot be satisfied. Although this argument does not rigorously prove that the corresponding entrained subset will disappear, since the calculations leading to the above equations require Kr'_{rms} to be small with respect to $\gamma_{\tilde{\omega}}$ (see the auxiliary material [30]), the first example in the next section illustrates that an entrained subset may indeed disappear with increasing K. Considering, however, the density function for the second example, a symmetry argument indicates that this does not apply as a general rule: if the presence of another entrained subset is required for this phenomenon, then the largest two entrained subsets in this example cannot disappear simultaneously with increasing K. They will both persist when K is increased, until they merge and form one entrained subset.

IV. EXAMPLES

A. Example 1

In a first example we consider the density function g_1 defined by

$$g_1(\omega) \triangleq \frac{1}{1.04\sqrt{\pi}}(e^{-\omega^2} + e^{-50^2(\omega-1.2)^2} + e^{-50^2(\omega+1.2)^2}),$$

for all ω in $\mathbb{R}$; see Fig. 1. Figure 2(a) shows simulation results for a system of $N=1000$ oscillators. In order to ascertain that the ω values are chosen according to g_1, we make sure that they are mapped to the values $\frac{1}{2N}, \frac{3}{2N}, \ldots, \frac{2N-1}{2N}$ by the cumulative distribution function associated with g_1. In Fig. 2(b) the intervals for ω corresponding to the entrained subsets are compared to analytical predictions. The analytical predictions are obtained by varying Kr_0, and for each value of Kr_0 (numerically) calculating K from (9a), and (numerically) solving (10a) and (10b) for $\gamma_{\tilde{\omega}}$ and $|R_R'(\gamma_{\tilde{\omega}})|$.

As is clearly visible in Fig. 2(b) for both simulations and analysis, the length of the interval corresponding to the entrained subset first increases but then decreases again with increasing K, until the entrained subset has disappeared.

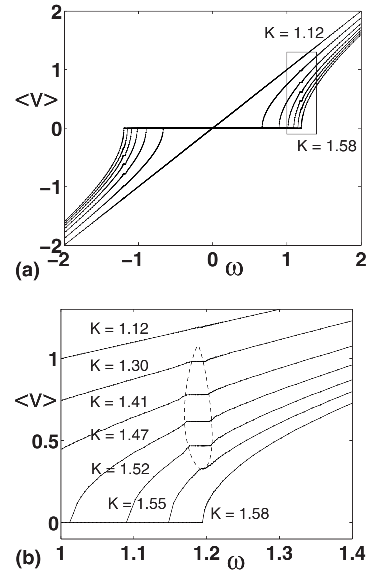

FIG. 2. Long-term average frequencies vs natural frequencies (determined by g_1) for different values of the coupling strength, resulting from a simulation with $N=1000$. (b) is obtained by zooming in on the rectangular region indicated (a). The dashed line shows an analytical prediction of the boundaries of the entrained subsets.

When K is increased further the oscillators are absorbed in the larger entrained subset, but before that happens there is a small window for K for which there is no entrainment for the corresponding oscillators.

This phenomenon, for which entrainment disappears with increasing coupling strength, also occurs in systems with a finite number of oscillators ([28], [27] pp. 46–47) in a similar way. The techniques used in the present paper cannot be applied to systems with a (small) finite number of oscillators since in our present approach oscillators in a small entrained subset constitute a small fraction of the population, while one oscillator in a finite population always constitutes a nonzero fraction of the population. The finite number of oscillators also results in time fluctuations of the order parameter [of the size $O(N^{-1/2})$] around the value in the system with an infinite number of oscillators [25], which would complicate a similar approach based on the frequency decomposition of the order parameter. (Of course, the larger the population, the better the system may be described as an infinite population.) Furthermore, the use of integrals in obtaining the self-consistency equation is more convenient than summation over a finite number of oscillators.

However, the intuitive explanation offered in [28] also applies to the present paper. In both cases, the entrainment breakup can be attributed to a differentiated attraction from one entrained subset on the oscillators in another entrained subset, tearing this latter entrained subset apart. The explanation for an infinite number of oscillators can be clarified as follows.

As can be seen in Fig. 2, the graph depicting the long-term average frequencies as a function of ω (the average frequency for $|\omega|>Kr_0$ can be approximated by $\gamma_\omega = \sqrt{\omega^2-Kr_0^2}$) has a large slope for drifting oscillators with ω

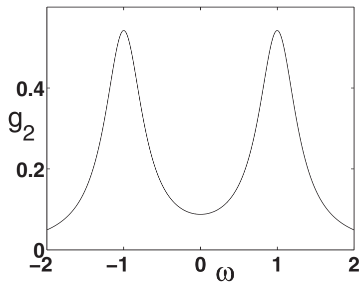

FIG. 3. Density function g_2.

values just outside the interval $[-Kr_0, Kr_0]$. This can be interpreted as a stronger attraction (in "frequency space") toward the entrained subset for oscillators closer to the entrained subset. This differential attraction tends to tear the smaller entrained subset apart, and this obstruction of entrainment is reflected in Fig. 2 by the contrast between the large slope in the neighborhood of the larger entrained subset and the zero slope within the smaller entrained subset.

B. Example 2

In a second example we illustrate the emergence of entrained subsets that cannot be accounted for by the distribution of the natural frequencies. In the previous example this phenomenon can only be observed after a substantial increase in the number of oscillators, since the associated intervals for ω are quite small due to choosing the extra peaks in g_1 sufficiently narrow for a good agreement of the simulations with the analysis. We will therefore consider a density function with broader peaks; the density function g_2 is defined by

$$g_2(\omega) \triangleq \frac{0.15}{\pi}\left(\frac{1}{0.09+(\omega-1)^2}+\frac{1}{0.09+(\omega+1)^2}\right),$$

for all ω in $\mathbb{R}$, and depicted in Fig. 3. Instead of considering g_2 as an even density function with no middle peak, our approach requires that one peak is the unperturbed density function and the other the perturbation. Of course the mathematical analysis cannot provide accurate quantitative results in this case, but the prediction regarding the existence of induction of entrained subsets is still valid.

In Fig. 4, which is obtained in a similar way as Fig. 2, only the largest two entrained subsets arise from peaks in the density function g_2. For different values of the coupling strength other entrained subsets can be distinguished, unaccounted for by the shape of g_2. For a fixed K, the differences in long-term average frequencies of the various entrained subsets are multiples of the average frequency difference between the largest two entrained subsets, in accordance with simulation results from [24,27]. This phenomenon disappears when K is increased further and the largest two entrained subsets merge, forming one entrained subset.

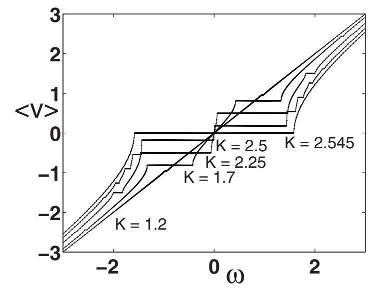

FIG. 4. Long-term average frequencies vs natural frequencies (determined by g_2) for different values of the coupling strength, resulting from a simulation with $N=1000$.

Under some extra assumptions the model (1) may represent an array of Josephson junctions [31]. In this context the corresponding steps in Fig. 4 are also referred to as Shapiro steps [29].

V. CONCLUSION

For a particular class of multimodal natural frequency distributions, the Kuramoto-Sakaguchi model with an infinite number of oscillators can be investigated analytically by considering a frequency decomposition of the complex order parameter. For each frequency component, a self-consistency equation can be formulated, with contributions from both the entrained oscillators and the drifting oscillators. The analytical results shed light on two phenomena that require the presence of multiple partially entrained subsets.

They predict that partial entrainment in a smaller subset may disappear with increasing coupling strength as a larger entrained subset grows and its border approaches the smaller subset. The numerical results show (in accordance with the simulations) that the transition is gradual: the size of the smaller subsets decreases and goes to zero as the coupling strength increases.

The results also establish that entrained subsets do not necessarily originate from high densities in the natural frequency distribution of the oscillators; entrainment may also be induced by resonances with other entrained subsets in a similar way as for the externally driven Kuramoto model. Our analysis shows that the average frequencies of these entrained subsets are linear combinations (with integer coefficients) of the average frequencies of other entrained subsets that are induced by high frequency densities.

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